Ch: Matrices

Use of the Inverse of a Matrix to solve a System of Linear Equations:

Let the system of equations be

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

These equations in the matrix notation can be written as

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

i.e. $AX = B$

where A is the co-efficient matrix $= \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

X is the columns matrix of the unknown $= \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

B is the column matrix consisting of the constant terms.

To solve the given linear equations means to solve the matrix equations $AX = B$ i.e to find X.

Theorem If $AX = B$ is a system of linear equations with $|A| \neq 0$, then $X = A^{-1}B$ is the unique solution.

Pf: The system of equations is

$$AX = B \longrightarrow (1)$$

$\because$ $|A| \neq 0$, A^{-1} exists.

Pre-multiply both sides of (1) by A^{-1}

$$A^{-1}(AX) = A^{-1}B$$

or $(A^{-1}A)X = A^{-1}B$

or $IX = A^{-1}B$

or $X = A^{-1}B$

which is the solution of equation (1).

Uniqueness

If possible, let X_1 and X_2 be two solutions of (1).

$\therefore$ $AX_1 = B \longrightarrow (2)$

and $AX_2 = B \longrightarrow (3)$

From (2) and (3), we get $AX_1 = AX_2$

Pre-multiplying both sides by A^{-1}

$$A^{-1}(AX_1) = A^{-1}(AX_2)$$

or $(A^{-1}A)X_1 = (A^{-1}A)X_2$

or $IX_1 = IX_2$

or $X_1 = X_2$.

Which shows that $AX = B$ has unique solution.

Examples Solve the following equations by matrices:

$$x + z = 7$$
$$2x + y = 7$$
$$3x + 2y + z = 17$$

Sol^n: Given equations are

$$x + 0.y + z = 7$$
$$2x + y + 0.z = 7$$
$$3x + 2y + z = 17.$$

In the matrix notation, we can write as

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 17 \end{bmatrix}$$

or $$AX = B.$$

where $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 7 \\ 7 \\ 17 \end{bmatrix}$.

Now $$|A| = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 2 & 2 & 0 \end{vmatrix} \quad R_3 \to R_3 - R_1$$

$$= \begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} \quad [\text{Expanding by col. } 3]$$

$$= 4 - 2 = 2 \neq 0.$$

$\Rightarrow$ A is non-singular.

$\therefore$ The above equations possess unique solution.

Unique solution is $X = A^{-1}B$.

To find A^{-1},

co-factor of $1 = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1-0 = 1$

co-factor of $0 = -\begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} = -2$

co-factor of $1 = \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 1$

co-factor of $2 = -\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} = 2$

co-factor of $1 = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} = -2$

co-factor of $0 = -\begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix} = -2$

co-factor of $3 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$

co-factor of $2 = -\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = 2$

co-factor of $1 = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1$

$\therefore$ adj A = transpose of the matrix $\begin{bmatrix} 1 & -2 & 1 \\ 2 & -2 & -2 \\ -1 & 2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 2 & -1 \\ -2 & -2 & 2 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj} A = \frac{1}{2}\begin{bmatrix} 1 & 2 & -1 \\ -2 & -2 & 2 \\ 1 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & 1 & -\frac{1}{2} \\ -1 & -1 & 1 \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix}$$

Using $X = A^{-1}B$, we get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 1 & -\frac{1}{2} \\ -1 & -1 & 1 \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 7 \\ 7 \\ 17 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{7}{2} + 7 - \frac{17}{2} \\ -7 - 7 + 17 \\ \frac{7}{2} - 7 + \frac{17}{2} \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

$\Rightarrow \quad x = 2, \quad y = 3, \quad z = 5.$

Practise Example

Solve the following equations by matrices:

$$2x - y + z = 3$$
$$x + 3y - 2z = 1$$
$$x + y + z = 6.$$

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