Ch: Matrices.

Rank of a Matrix:-

Let A be any $m \times n$ matrix. It has square sub-matrices of different orders. The determinants of these square sub-matrices are called minors of A. If all minors of order $(r+1)$ are zero but there is at least one non-zero minor of order r, then r is called the rank of A. Symbolically, rank of A is written as $\rho(A) = r$.

From the definition of the rank of a matrix A, it follows that:

i) If A is a null matrix, then $\rho(A) = 0$.

ii) If A is not a null matrix, then $\rho(A) \geqslant 1$.

iii) If A is a non-singular $n \times n$ matrix, then $\rho(A) = n$.

[$\because |A| \neq 0$ is the largest minor of A.

If I_n is the $n \times n$ unit matrix, then

$$|I_n| = 1 \neq 0 \Rightarrow \rho(I_n) = n.]$$

iv) If all minors of order r are equal to zero, then $\rho(A) < r$.

Different methods to determine the rank of a matrix A:-

i) Start with the highest order minor (or minors) of A. Let their order be r. If any one of them is non-zero, then $\rho(A) = r$.

If all of them are zero, start with minors of next order $(r-1)$ and so on till you get a non-zero minor. The order of that minor is the rank of A.

This method usually involves a lot of computational work since we have to evaluate several determinants.

ii) If A is an $m\times n$ matrix and by a series of elementary (row or column or both) operations, it can be put into one of the following forms (called normal forms):

$$\begin{bmatrix} I_r & \vdots & 0 \\ \cdots & & \cdots \\ 0 & & 0 \end{bmatrix}, \begin{bmatrix} I_r \\ 0 \end{bmatrix}, \begin{bmatrix} I_r & \vdots & 0 \end{bmatrix}, \begin{bmatrix} I_r \end{bmatrix}$$ where I_r is the unit matrix of order r.

Since the rank of a matrix is not changed as a result of elementary transformations, it follows that

$$\rho(A) = r \quad [\because r^{th} \text{ order minor } |I_r| = 1 \neq 0.$$

Examples

Find the rank of the matrix by normal form or canonical form

$$\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

Soln :- Here $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ is a 4×4 matrix.

$$\therefore \rho(A) \leq 4.$$

$$\Rightarrow A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix} \quad \text{operating } R_{12} \text{ i.e. } R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 5 & 3 & 7 \\ 3 & 4 & 9 & 10 \\ 6 & 9 & 12 & 17 \end{bmatrix} \quad \begin{array}{l} C_2 \to C_2 + C_1 \\ C_3 \to C_3 + 2C_1 \\ C_4 \to C_4 + 4C_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 6R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 1 & -6 & -3 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - R_3 \\ R_4 \to R_4 - 2R_3 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & 33 & 22 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} C_3 \to C_3 + 6C_2 \\ C_4 \to C_4 + 3C_2. \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} R_3 \to R_3 - 4R_2 \\ R_4 \to R_4 - R_2. \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad C_3 \to \frac{1}{33} C_3.$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad C_4 \to C_4 - 22 C_3.$$

$$= \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore \rho(A) = 3.$

Practise Examples

Find the rank of the matrix by normal forms:

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}.$$

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