ECO-HC-2026- Mathematical Methods in Economics- II.

Linear Algebra:

Solutions of simultaneous equations by using Cramer's rule:-

Determinant of a Square Matrix:-

Defn: Let A be any square matrix of the field F, then the determinant of the square matrix is denoted by |A|.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a square matrix of order 3 (≥ 1) over the field F.

Then |A| is defined as below:

$$|A| = \left| \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \right|$$

$$= a_{11} \left| \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \right| - a_{12} \left| \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} \right| + a_{13} \left| \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \right|$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}).$$

Note:- 1. If the elements of a matrix A are numbers, then so are of |A|.

2. Determinant cannot be associated with non-square matrices.

3. The number of terms in the expansion of a square matrix A of order n is n.

Minors and Co-factors :-

i) The minor of an element in a determinant is that which is obtained by leaving the row and column in which that element occurs.

ii) The co-factor of an element in a determinant is its minor with proper sign.

Notations :- We denote the minor and co-factor of a_{ij} in a determinant by M_{ij} and A_{ij} respectively.

Note :- $A_{ij} = (-1)^{i+j} M_{ij}$ i.e $= (-1)^{\text{row}+\text{column}} M_{ij}$

Thus if $A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then

(i) $M_{11} = \text{Minor of } a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

$A_{11} = \text{Co-factor of } a_{11} = (-1)^{1+1} M_{11} = M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$.

ii) $M_{12} = \text{Minor of } a_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

$A_{12} = \text{Co-factor of } a_{12} = (-1)^{1+2} M_{12} = -M_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

and so on.

Defns

i) Linear Equations :- The equations of the form:
$a_1x + b_1y + c_1z + d_1 = 0$, $a_2x + b_2y + c_2z + d_2 = 0$, $a_3x + b_3y + c_3z + d_3 = 0$ are known as linear equations in x, y, z.

ii) A system of equations is said to be consistent if it has at least one solution.

Solution of a system of three linear equations by Cramer's rule:-

Let the system of linear equations are

$$a_1x + b_1y + c_1z + d_1 = 0$$
$$a_2x + b_2y + c_2z + d_2 = 0$$
$$a_3x + b_3y + c_3z + d_3 = 0.$$

Then by Cramer's rule, we have

$$\frac{x}{\Delta_1} = \frac{-y}{\Delta_2} = \frac{z}{\Delta_3} = \frac{-1}{\Delta}$$

where $\Delta_1 = \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix}$

$\Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$ and $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

i.e. $\Delta_1, \Delta_2, \Delta_3$ are determinants formed by the coefficients by omitting those of x, y, z respectively.

Example:- With the help of determinants (by Cramer's rule), solve the following equations:

$$x+y+z=7,\quad x+2y+3z=16,\quad x+3y+4z=22.$$

Soln:- The given equations are

$$x+y+z-7=0 \longrightarrow (1)$$
$$x+2y+3z-16=0 \longrightarrow (2)$$
and $$x+3y+4z-22=0 \longrightarrow (3)$$

By Cramer's rule, we have

$$\frac{x}{\Delta_1} = \frac{-y}{\Delta_2} = \frac{z}{\Delta_3} = \frac{-1}{\Delta} \longrightarrow (4)$$

where $\Delta_1 = \begin{vmatrix} 1 & 1 & -7 \\ 2 & 3 & -16 \\ 3 & 4 & -22 \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} 1 & 1 & -7 \\ 1 & 3 & -16 \\ 1 & 4 & -22 \end{vmatrix}$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & -7 \\ 1 & 2 & -16 \\ 1 & 3 & -22 \end{vmatrix}, \quad \Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{vmatrix}.$$

Now, $\Delta_1 = \begin{vmatrix} 1 & 1 & -7 \\ 2 & 3 & -16 \\ 3 & 4 & -22 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 5 \\ 3 & 1 & 6 \end{vmatrix}$ $\quad C_2 \to C_2 - C_1$, $C_3 \to C_3 + 7C_1$

$$= 1\begin{vmatrix} 1 & 5 \\ 1 & 6 \end{vmatrix} = 1(6-5) = 1.$$

$\Delta_2 = \begin{vmatrix} 1 & 1 & -7 \\ 1 & 3 & -16 \\ 1 & 4 & -22 \end{vmatrix} = \begin{vmatrix} 1 & 1 & -7 \\ 0 & 2 & -9 \\ 0 & 1 & -6 \end{vmatrix}$ $\quad R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_2$.

$$= 1\begin{vmatrix} 2 & -9 \\ 1 & -6 \end{vmatrix} = 1(-12+9) = -3.$$

$\Delta_3 = \begin{vmatrix} 1 & 1 & -7 \\ 1 & 2 & -16 \\ 1 & 3 & -22 \end{vmatrix} = \begin{vmatrix} 1 & 1 & -7 \\ 0 & 1 & -9 \\ 0 & 1 & -6 \end{vmatrix}$ $\quad R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_2$.

$$= 1\begin{vmatrix} 1 & -9 \\ 1 & -6 \end{vmatrix} = 1(-6+9) = 3.$$

and $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{vmatrix}$ $\quad R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_2$.

$$= 1\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1(1-2) = -1.$$

From (4), $\dfrac{x}{1} = \dfrac{-y}{-3} = \dfrac{z}{3} = \dfrac{-1}{-1}$

$\therefore \dfrac{x}{1} = \dfrac{-1}{-1} \Rightarrow x = 1$, $\quad \dfrac{-y}{-3} = \dfrac{-1}{-1} \Rightarrow y = 3$

$\dfrac{z}{3} = \dfrac{-1}{-1} \Rightarrow z = 3.$

Hence $x = 1$, $y = 3$, $z = 3$. #

Practise Example :-

Solve the following equations, with the help of determinants (by Cramer's rule) :-

$x + y + z = 2$, $x + 2y + 3z = 1$, $3x + y - 5z = 4$.

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