ECO-HC-2026- Mathematical Methods in Economics-II

1. Linear Algebra:

A] Adjoint and Inverse of a Matrix:

Minor of a Matrix:

Def.: The determinant of every square sub-matrix of the matrix A is said to be the minor of A.

For Example: If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, then minors of A are

$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$, $\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$, $\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix}$; etc.

Transpose of a Matrix:

Def: The matrix obtained from any given matrix A by interchanging its rows and columns is called its transpose.

Notation: The transpose of a matrix A is denoted by A' or A^T.

For Example: If $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 2 & 5 \end{bmatrix}$, then $A' = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 2 \\ 4 & 1 & 5 \end{bmatrix}$

Adjoint of a Square Matrix:

Def: The adjoint of a square matrix is the transpose of the matrix obtained by replacing each element of A by its co-factor in $|A|$.

Thus, if $A = [a_{ij}]$ be any square matrix of the type $n \times n$, then the transpose $[A_{ij}]'$ of the matrix $[A_{ij}]$ formed by the co-factors of a_{ij} in the determinant $|A|$ is called the adjoint of A and is written as adj. A.

How to find adj. A ?

(I) Replace each element of A by its co-factor in $|A|$.

(II). Take the transpose.

Example 1. Find the adjoint of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$.

Soln:- Here $|A| = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 2 & -1 \\ 3 & 1 & -1 \end{vmatrix} \quad C_3 \to C_3 - 2C_2$

$= -1(-1+3) = -2.$

Now,

A_{11} = co-factor of $0 = \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = 2-3 = -1.$

A_{12} = co-factor of $1 = -\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = -(1-9) = 8.$

A_{13} = co-factor of $2 = \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 1-6 = -5.$

A_{21} = co-factor of $1 = -\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -(1-2) = 1.$

A_{22} = co-factor of $2 = \begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} = 0-6 = -6.$

A_{23} = co-factor of $3 = -\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} = -(0-3) = 3.$

A_{31} = co-factor of $3 = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3-4 = -1.$

A_{32} = co-factor of $1 = -\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} = -(0-2) = 2.$

A_{33} = co-factor of $1 = \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = 0-1 = -1.$

$\therefore$ adj. A = transpose of $\begin{bmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$

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Practise Example

Calculate the adjoint of the following matrix:

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 7 \\ 4 & 7 & 9 \end{bmatrix}.$$

Inverse of a Matrix:

Def.: Any n-rowed square matrix A is said to be invertible if there exists an n-rowed matrix B such that

$AB = BA = I_n$, where I_n is a unit matrix of order n.

B is said to be the inverse of A.

Notation: The inverse of A is denoted by A^{-1}.

Remarks: 1. A non-square matrix does not possess any inverse.

II. If A is the inverse of B, then B is the inverse of A.

Singular and Non-Singular Matrices:

Def: A square matrix A is said to be singular if $|A| = 0$ and non-singular if $|A| \neq 0$.

S: The necessary and sufficient condition for a square A to be invertible is that $|A| \neq 0$.

Cor.:- $A^{-1} = \dfrac{\text{adj. } A}{|A|}$ when $|A| \neq 0$.

Example: Find the inverse of $A = \begin{bmatrix} 3 & 5 & 7 \\ 2 & -3 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

Soln:- Here $|A| = \begin{vmatrix} 3 & 5 & 7 \\ 2 & -3 & 1 \\ 1 & 1 & 2 \end{vmatrix}$

$= \begin{vmatrix} 0 & 2 & 1 \\ 0 & -5 & -3 \\ 1 & 1 & 2 \end{vmatrix}$ $R_1 \to R_1 - 3R_3$, $R_2 \to R_2 - 2R_3$.

$$= 1\begin{vmatrix} 2 & 1 \\ -5 & -3 \end{vmatrix} \quad [\text{Expanding by col. } 1].$$

$$= 1(-6+5) = -1.$$

Since $|A| = -1 \neq 0$ $\therefore$ A^{-1} exists.

Now, $A_{11} =$ co-factor of $3 = \begin{vmatrix} -3 & 1 \\ 1 & 2 \end{vmatrix} = -6-1 = -7.$

$A_{12} =$ co-factor of $5 = -\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = -(4-1) = -3.$

$A_{13} =$ co-factor of $7 = \begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} = 2+3 = 5$

$A_{21} =$ co-factor of $2 = -\begin{vmatrix} 5 & 7 \\ 1 & 2 \end{vmatrix} = -(10-7) = -3.$

$A_{22} =$ co-factor of $3 = \begin{vmatrix} 3 & 7 \\ 1 & 2 \end{vmatrix} = 6-7 = -1.$

$A_{23} =$ co-factor of $1 = -\begin{vmatrix} 3 & 5 \\ 1 & 1 \end{vmatrix} = -(3-5) = 2.$

$A_{31} =$ co-factor of $1 = \begin{vmatrix} 5 & 7 \\ -3 & 1 \end{vmatrix} = 5+21 = 26.$

$A_{32} =$ co-factor of $1 = -\begin{vmatrix} 3 & 7 \\ 2 & 1 \end{vmatrix} = -(3-14) = 11$

$A_{33} =$ co-factor of $2 = \begin{vmatrix} 3 & 5 \\ 2 & -3 \end{vmatrix} = -9-10 = -19.$

$$\therefore \text{adj. } A = \text{transpose of } \begin{bmatrix} -7 & -3 & 5 \\ -3 & -1 & 2 \\ 26 & 11 & -19 \end{bmatrix} = \begin{bmatrix} -7 & -3 & 26 \\ -3 & -1 & 11 \\ 5 & 2 & -19 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{\text{adj. } A}{|A|} = \frac{1}{-1}\begin{bmatrix} -7 & -3 & 26 \\ -3 & -1 & 11 \\ 5 & 2 & -19 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 3 & -26 \\ 3 & 1 & -11 \\ -5 & -2 & 19 \end{bmatrix}.$$

Practise Example :-

Find the inverse of the following matrix

$$\begin{bmatrix} 0 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}.$$

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