De Moivre's Theorem. Statement.

(i) If n is any integer (+ ve, – ve or zero), then

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

(M.D.U. 1991 S, 92 ; Meerut 1985, 89)

and (ii) If n is a fraction (+ ve or – ve), then one of the values of

$$(\cos\theta + i\sin\theta)^n \text{ is } \cos n\theta + i\sin n\theta.$$

(M.D.U. 1989 ; D.U. 1986, 88, 92, 94)

Proof : Case I. *When n is a positive integer.*

We shall prove the theorem by induction on n.

When $n = 1$, the theorem becomes

$$(\cos\theta + i\sin\theta)^1 = \cos 1\theta + i\sin 1\theta$$

$$\Rightarrow \quad \cos\theta + i\sin\theta = \cos\theta + i\sin\theta$$

which is true.

Let us suppose the theorem is true for $n = m$.

i.e., let $(\cos\theta + i\sin\theta)^m = \cos m\theta + i\sin m\theta$ ...(1)

Now $(\cos\theta + i\sin\theta)^{m+1}$

$= (\cos\theta + i\sin\theta)^m (\cos\theta + i\sin\theta)$

$= (\cos m\theta + i\sin m\theta)(\cos\theta + i\sin\theta)$ [using (1)]

$= (\cos m\theta\cos\theta - \sin m\theta\sin\theta) + i(\sin m\theta\cos\theta + \cos m\theta\sin\theta)$

$= \cos(m\theta + \theta) + i\sin(m\theta + \theta)$

$= \cos(m+1)\theta + i\sin(m+1)\theta$

$\Rightarrow$ The theorem is true for $n = m + 1$

Hence by the Principle of Mathematical Induction, the theorem is true for all positive integers n.

Case II. *When n is a negative integer.*

Let $n = -m$, where m is a positive integer

$\therefore \quad (\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m}$

$$= \frac{1}{(\cos\theta + i\sin\theta)^m} = \frac{1}{\cos m\theta + i\sin m\theta} \quad \text{[by Case I]}$$

$$= \frac{1}{\cos m\theta + i\sin m\theta} \times \frac{\cos m\theta - i\sin m\theta}{\cos m\theta - i\sin m\theta}$$

$$= \frac{\cos m\theta - i \sin m\theta}{\cos^2 m\theta - i^2 \sin^2 m\theta} = \frac{\cos m\theta - i \sin m\theta}{\cos^2 m\theta + \sin^2 m\theta} \quad [\because \quad i^2 = -1]$$

$$= \cos m\theta - i \sin m\theta = \cos (-m)\theta + i \sin (-m)\theta$$

$$[\because \quad \cos(-\theta) = \cos\theta \; ; \; \sin(-\theta) = -\sin\theta]$$

$$= \cos n\theta + i \sin n\theta \qquad | \because \quad -m = n$$

Case III. *When n is a fraction, positive or negative.*

Let $n = \frac{p}{q}$ *where* **q is a positive integer** and p is any integer, positive or negative, **p and q have no common factor** (*i.e.*, p and q are prime to each other). It follows from case I, that

$$\left(\cos\frac{\theta}{q} + i \sin\frac{\theta}{q}\right)^q = \cos q \cdot \frac{\theta}{q} + i \sin q \cdot \frac{\theta}{q} = \cos\theta + i \sin\theta$$

Taking qth root of both sides.

$\cos\frac{\theta}{q} + i \sin\frac{\theta}{q}$ is **one of the values of** $(\cos\theta + i \sin\theta)^{1/q}$

Raising to pth power,

$\therefore \quad \left(\cos\frac{\theta}{q} + i \sin\frac{\theta}{q}\right)^p$ **is one of the values of** $(\cos\theta + i \sin\theta)^{p/q}$

Since p may be a + ve or – ve integer,

$\therefore$ by cases I and II

$\cos\frac{p}{q}\theta + i \sin\frac{p}{q}\theta$ is **one of the values of** $(\cos\theta + i \sin\theta)^{p/q}$

$\Rightarrow \quad \cos n\theta + i \sin n\theta$ is **one of the values of** $(\cos\theta + i \sin\theta)^n$

Hence De Moivre's Theorem is completely established.

Cor. 1. $(\cos\theta + i \sin\theta)^{-n} = \cos(-n\theta) + i \sin(-n\theta)$
$= \cos n\theta - i \sin n\theta$

Cor. 2. $(\cos\theta - i \sin\theta)^n = [\cos(-\theta) + i \sin(-\theta)]^n$
$= \cos(-n\theta) + i \sin(-n\theta)$
$= \cos n\theta - i \sin n\theta$

Cor. 3. $(\cos\theta - i \sin\theta)^{-n} = [\cos(-\theta) + i \sin(-\theta)]^{-n}$
$= \cos n\theta + i \sin n\theta$

Cor. 4. $\frac{1}{\cos\theta + i \sin\theta} = (\cos\theta + i \sin\theta)^{-1} = \cos\theta - i \sin\theta$

Caution. For the application of De Moivre's Theorem

1. Real part must be with cos and imaginary part with sin *i.e.*, De Moivre's Theorem can not be directly applied to

$$(\sin\theta + i \cos\theta)^n$$

Procedure :

$$(\sin\theta + i\cos\theta)^n = \left[\cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)\right]^n$$

[Note this step]

$$= \cos n\left(\frac{\pi}{2} - \theta\right) + i\sin n\left(\frac{\pi}{2} - \theta\right)$$

2. The angle with sin and cos must be the same *i.e.*, De Moivre's Theorem cannot be applied to $(\cos\alpha + i\sin\beta)^n$.

☞**Note.** $(\text{cis}\,\theta_1)(\text{cis}\,\theta_2)\,.....\,(\text{cis}\,\theta_n) = \text{cis}\,(\theta_1 + \theta_2 + ... + \theta_n)$

Example 1. *Simplify :* $\dfrac{(\cos 3\theta + i\sin 3\theta)^5 (\cos\theta - i\sin\theta)^3}{(\cos 5\theta + i\sin 5\theta)^7 (\cos 2\theta - i\sin 2\theta)^5}$.

Sol. $\dfrac{(\cos 3\theta + i\sin 3\theta)^5 (\cos\theta - i\sin\theta)^3}{(\cos 5\theta + i\sin 5\theta)^7 (\cos 2\theta - i\sin 2\theta)^5}$

$$= \frac{[(\cos\theta + i\sin\theta)^3]^5\,[(\cos\theta + i\sin\theta)^{-1}]^3}{[(\cos\theta + i\sin\theta)^5]^7\,[(\cos\theta + i\sin\theta)^{-2}]^5}$$

$$= \frac{(\cos\theta + i\sin\theta)^{15}(\cos\theta + i\sin\theta)^{-3}}{(\cos\theta + i\sin\theta)^{35}(\cos\theta + i\sin\theta)^{-10}}$$

$$= (\cos\theta + i\sin\theta)^{15 - 3 - 35 + 10}$$

$$= (\cos\theta + i\sin\theta)^{-13} = \cos 13\theta - i\sin 13\theta.$$

Example 2. *Prove that :*

$$\frac{(\cos 3\theta + i\sin 3\theta)^5 (\cos 2\theta - i\sin 2\theta)^3}{(\cos 4\theta + i\sin 4\theta)^{-9} (\cos 5\theta + i\sin 5\theta)^9} = 1.$$

Sol. L.H.S. $= \dfrac{[(\cos\theta + i\sin\theta)^3]^5\,[(\cos\theta + i\sin\theta)^{-2}]^3}{[(\cos\theta + i\sin\theta)^4]^{-9}\,[(\cos\theta + i\sin\theta)^5]^9}$

$$= \frac{(\cos\theta + i\sin\theta)^{15}(\cos\theta + i\sin\theta)^{-6}}{(\cos\theta + i\sin\theta)^{-36}(\cos\theta + i\sin\theta)^{45}}$$

$$= (\cos\theta + i\sin\theta)^{15 - 6 + 36 - 45} = (\cos\theta + i\sin\theta)^0 = 1.$$

Example 3. *Simplify :*

$$\frac{(\cos\theta + i\sin\theta)^3 (\cos\theta - i\sin\theta)^5}{(\cos 2\theta + i\sin 2\theta)^4}.$$

Sol. Please try yourself.

[Ans. $\cos 10\theta - i\sin 10\theta$**]**

Example 4. *Simplify :* $\left(\dfrac{\cos\theta + i\sin\theta}{\sin\theta + i\cos\theta}\right)^4$.

Sol. $\left[\dfrac{\cos\theta + i\sin\theta}{\sin\theta + i\cos\theta}\right]^4 = \dfrac{(\cos\theta + i\sin\theta)^4}{\left[\cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)\right]^4}$

$$= \frac{(\cos\theta + i\sin\theta)^4}{\cos 4\left(\frac{\pi}{2} - \theta\right) + i\sin 4\left(\frac{\pi}{2} - \theta\right)}$$

$$= \frac{(\cos\theta + i\sin\theta)^4}{\cos(2\pi - 4\theta) + i\sin(2\pi - 4\theta)}$$

$$= \frac{(\cos\theta + i\sin\theta)^4}{\cos 4\theta - i\sin 4\theta} = \frac{(\cos\theta + i\sin\theta)^4}{(\cos\theta + i\sin\theta)^{-4}}$$

$$= (\cos\theta + i\sin\theta)^8$$

$$= \cos 8\theta + i\sin 8\theta.$$