Sequential Criterion for limits:

We will now look at a very important theorem known as "The Sequential Criterion for a limit which merges the concepts of the limit of a function f at a cluster point c from A with regards to sequences (a_n) from A that converge to c.

Theorem (The Sequential Criterion for a limit of a function):- Let $f: A \to \mathbb{R}$ be a function and let c be a cluster point A. Then $\lim_{x \to c} f(x) = L$ if and only if for all sequences (a_n) from the domain A where $a_n \neq c \ \forall n \in \mathbb{N}$ and $\lim_{n \to \infty} a_n = c$ then $\lim_{n \to \infty} f(a_n) = L$.

Proof: Suppose that $\lim_{x \to c} f(x) = L$, and let (a_n) be a sequence in A such that $a_n \neq c \ \forall n \in \mathbb{N}$ such that $\lim_{n \to \infty} a_n = c$. We thus want to show that $\lim_{n \to \infty} f(a_n) = L$.

Let $\epsilon > 0$, We are given that $\lim_{x \to c} f(x) = L$ and so for $\epsilon > 0$ there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x - c| < \delta$ then we have that $|f(x) - L| < \epsilon$.

Now since $\delta > 0$, since we have that $\lim_{n \to \infty} a_n = c$ then there exists an $N \in \mathbb{N}$ such that if $n \geq N$ then $|a_n - c| < \delta$. Therefore $a_n \in V_\delta(c) \cap A$.

Therefore it must be that $|f(a_n)-L|<\epsilon$, in other words, $\forall\, n \geq N$ we have that $|f(a_n)-L|<\epsilon$ and so $\lim_{n\to\infty} f(a_n) = L$.

Suppose that for all (a_n) in A such that $a_n \neq c$ $\forall n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = c$, we have that $\lim_{n\to\infty} f(a_n) = L$.

we want to show that $\lim_{x\to c} f(x) = L$.

Suppose not, in other words, suppose that $\exists\, \epsilon_0 > 0$ such that $\forall\, \delta > 0$ the $\exists\, x_\delta \in A \cap V_\delta(c) \setminus \{c\}$ such that $|f(x_\delta) - L| \geq \epsilon_0$. Let $\delta_n = \frac{1}{n}$. Then there exists $x_{\delta_n} = a_n \in A \cap V_{\delta_n}(c) \setminus \{c\}$, in other words,

$0 < |a_n - c| < \frac{1}{n}$ and $\lim_{n\to\infty} a_n = c$. However;

$|f(a_n) - L| \geq \epsilon_0$ so $\lim_{n\to\infty} f(a_n) \neq L$, a Contradition.

Therefor $\lim_{x\to c} f(x) = L$.

<u>Limit of a function</u>:- In mathematics, the limit of a function is a fundamental Concept in Calculus and analysis concerning the behavior of that function near a particular input. Formal definitions, first devised in the early 19th Century, are given below.

A limit tells us the value that a function approaches as that function's inputs get closer and closer to some number.