Solution of equations using trigonometry and De Moivre's theorem:

Ex-1 If $x_r = \cos\frac{\pi}{2^r} + i\sin\frac{\pi}{2^r}$, prove that

$$x_1 x_2 x_3 \cdots \infty = -1.$$

Solⁿ:

$$x_r = \cos\frac{\pi}{2^r} + i\sin\frac{\pi}{2^r}$$

Putting $r = 1, 2, 3 \ldots$ we have

$x_1 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$, $x_2 = \cos\frac{\pi}{2^2} + i\sin\frac{\pi}{2^2}$

$x_3 = \cos\frac{\pi}{2^3} + i\sin\frac{\pi}{2^3}$ and so on.

$\therefore x_1 x_2 x_3 \cdots \infty$

$$= \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\left(\cos\frac{\pi}{2^2} + i\sin\frac{\pi}{2^2}\right)\left(\cos\frac{\pi}{2^3} + i\sin\frac{\pi}{2^3}\right)\cdots \text{ to } \infty$$

$$= \cos\left(\frac{\pi}{2} + \frac{\pi}{2^2} + \frac{\pi}{2^3} + \cdots \text{ to } \infty\right) + i\sin\left(\frac{\pi}{2} + \frac{\pi}{2^2} + \frac{\pi}{2^3} + \cdots \text{ to } \infty\right)$$

[By De Moivre's Thm.]

$$= \cos\frac{\pi/2}{1-\frac{1}{2}} + i\sin\frac{\pi/2}{1-\frac{1}{2}}$$

$$= \cos\pi + i\sin\pi$$

$$= -1 \quad [\because \cos\pi = -1,\ \sin\pi = 0].$$

Note: [$\frac{\pi}{2} + \frac{\pi}{2^2} + \frac{\pi}{2^3} + \cdots$ to ∞ is a G.P. Series.

Sum to infinity of G.P. Sum $= \frac{a}{1-r} = \frac{\frac{\pi}{2}}{1-\frac{1}{2}} = \pi$.]
where a is the first term and common ratio $r<1$

$r = \frac{\frac{\pi}{2^2}}{\frac{\pi}{2}} = \frac{1}{2}$ and $a = \frac{\pi}{2}$

Ex-2 If $x + \frac{1}{x} = 2\cos\theta$, prove that

$$x^n + \frac{1}{x^n} = 2\cos n\theta.$$

Solⁿ:

$$x + \frac{1}{x} = 2\cos\theta.$$

$$\Rightarrow x^2 + 1 = 2x\cos\theta.$$

$$\Rightarrow x^2 - 2x\cos\theta + 1 = 0$$

$\Rightarrow x^2 - 2x\cos\theta + \cos^2\theta + \sin^2\theta = 0.$

$\Rightarrow (x-\cos\theta)^2 = -\sin^2\theta.$

$\Rightarrow x - \cos\theta = \pm i\sin\theta. \quad | \because i^2 = -1.$

$\Rightarrow x = \cos\theta \pm i\sin\theta.$

$$\therefore x^n + \frac{1}{x^n} = x^n + x^{-n}$$
$$= (\cos\theta \pm i\sin\theta)^n + (\cos\theta \pm i\sin\theta)^{-n}$$
$$= (\cos n\theta \pm i\sin n\theta) + (\cos n\theta \mp i\sin n\theta)$$

[Upper signs to go together].

$$= 2\cos n\theta.$$

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Ex-3 If $x + \frac{1}{x} = 2\cos\theta$, $y + \frac{1}{y} = 2\cos\phi$, Prove that one of the values of $\frac{x^m}{y^n} + \frac{y^n}{x^m}$ is $2\cos(m\theta - n\phi)$.

Solⁿ: $x + \frac{1}{x} = 2\cos\theta$

$\Rightarrow x^2 + 1 = 2x\cos\theta \Rightarrow x^2 - 2x\cos\theta + \cos^2\theta + \sin^2\theta = 0.$

$\Rightarrow (x-\cos\theta)^2 = -\sin^2\theta \Rightarrow x - \cos\theta = \pm i\sin\theta.$

$\Rightarrow x = \cos\theta \pm i\sin\theta.$

$\therefore$ One of the values of x is $\cos\theta + i\sin\theta$.

Similarly one of the values of y is $\cos\phi + i\sin\phi$.

One of the values of $\frac{x^m}{y^n} = \frac{(\cos\theta + i\sin\theta)^m}{(\cos\phi + i\sin\phi)^n} = \frac{\cos m\theta + i\sin m\theta}{\cos n\phi + i\sin n\phi}$

$$= \cos(m\theta - n\phi) + i\sin(m\theta - n\phi).$$

One of the values of $\frac{y^n}{x^m}$ i.e. $\left[\frac{x^m}{y^n}\right]^{-1}$

$$= \left[\cos(m\theta - n\phi) + i\sin(m\theta - n\phi)\right]^{-1}$$
$$= \cos(m\theta - n\phi) - i\sin(m\theta - n\phi).$$

Hence one of the value of $\frac{x^m}{y^n} + \frac{y^n}{x^m}$ is $2\cos(m\theta - n\phi)$.

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Note 1. $\frac{1}{ab} = (ab)^{-1}$.

2. $(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) = \cos(\theta+\phi) + i\sin(\theta+\phi)$.

Ex-4 Apply De Moivre's Theorem to find an equation whose roots are the n^{th} powers of the roots of the equation $x^2 - 2x\cos\theta + 1 = 0$.

Soln:- Given equation is

$$x^2 - 2x\cos\theta + 1 = 0.$$

$$\therefore x = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \cos\theta \pm \sqrt{\cos^2\theta - 1}$$

$$= \cos\theta \pm \sqrt{-\sin^2\theta} = \cos\theta \pm i\sin\theta.$$

$\therefore$ Roots of given equation are $\cos\theta + i\sin\theta$ and $\cos\theta - i\sin\theta$.

Roots of required equation are

$$(\cos\theta + i\sin\theta)^n \text{ and } (\cos\theta - i\sin\theta)^n.$$

i.e. $\cos n\theta + i\sin n\theta$ and $\cos n\theta - i\sin n\theta$.

S = Sum of new roots = $2\cos n\theta$.

P = Product of new roots =

$$= (\cos n\theta + i\sin n\theta)(\cos n\theta - i\sin n\theta)$$

$$= \cos^2 n\theta + \sin^2 n\theta = 1.$$

$\therefore$ Required equation is

$$x^2 - Sx + P = 0.$$

i.e $x^2 - 2x\cos n\theta + 1 = 0$.