Evaluation of the roots.
Cubic equation

88, 90

Transformation of the Cubic

Let us consider the eqn $ax^3+3bx^2+3cx+d=0$

Let us put $x=t+h$,

The eqn reduces to

$\therefore$ $a(t+h)^3+3b(t+h)^2+3c(t+h)+d=0$.

$\Rightarrow a(t^3+3t^2h+3th^2+h^3)+3b(t^2+2th+h^2)+3ct+3ch+d=0$.

$\Rightarrow at^3+t^2(3ah+3b)+t(3ah^2+6bh+3c)+ah^3+3bh^2+3ch+d=0$.

Let us choose h in such a way that

$$ah+b=0$$
$$\Rightarrow h=-\frac{b}{a}$$

Hence the eqn reduces to

$$at^3+t\left(3a.\frac{b^2}{a^2}-6b.\frac{b}{a}+3c\right)-a.\frac{b^3}{a^3}+3b.\frac{b^2}{a^2}-3c.\frac{b}{a}+d=0$$

$$\Rightarrow at^3+t\left(3\frac{b^2}{a}-6\frac{b^2}{a}+3c\right)-\frac{b^3}{a^2}+\frac{3b^3}{a^2}-3\frac{bc}{a}+d=0$$

$$\Rightarrow at^3+t\left(3c-3\frac{b^2}{a}\right)+2\frac{b^3}{a^2}-3\frac{bc}{a}+d=0.$$

$\Rightarrow t^3+3Ht+G=0$ where $\longrightarrow$ (A).

$$H=\frac{c-\frac{b^2}{a}}{a}=\frac{\cancel{ac}-\frac{b^2}{a^2}}{}\ \frac{ac-b^2}{a^2}$$

$$G=\frac{\frac{2b^3}{a^2}-\frac{3bc}{a}+d}{a}=\frac{a^2d-3abc+2b^3}{a^3}$$

Cardon's Method of Soln:

Let $p^{\frac{1}{3}}+q^{\frac{1}{3}}$ be the roots of (A).

i.e $t=p^{\frac{1}{3}}+q^{\frac{1}{3}}$

$$\Rightarrow t^3=p+q+3p^{\frac{1}{3}}.q^{\frac{1}{3}}.t$$

$$\Rightarrow t^3-3p^{\frac{1}{3}}q^{\frac{1}{3}}t-(p+q)=0\rightarrow \text{(B)}$$

Comparing (A) and (B), $p^{\frac{1}{3}}q^{\frac{1}{3}}=-H$, $p+q=-G$.

Hence p and q are the roots of the eqn.

$$z^2+Gz-H^3=0$$

which gives $p=\frac{1}{2}(-G+\sqrt{G^2+4H^3})$

and $q=\frac{1}{2}(-G-\sqrt{G^2+4H^3})$

As cube root of p has three values, we get the value of z on substitution of these. #

Ex ⑥. Solve $x^3 - 21x - 344 = 0$. $\longrightarrow$ ①

Soln:- Let $x = p^{\frac{1}{3}} + q^{\frac{1}{3}}$

$\Rightarrow x^3 = p + q + 3p^{\frac{1}{3}}q^{\frac{1}{3}} \cdot x$.

$\Rightarrow x^3 - 3p^{\frac{1}{3}}q^{\frac{1}{3}}x - (p+q) = 0 \longrightarrow$ ②

Since ① and ② are identical

$\therefore -3p^{\frac{1}{3}}q^{\frac{1}{3}} = -21$ and $p+q = 344 \longrightarrow$ ③

$\Rightarrow p^{\frac{1}{3}}q^{\frac{1}{3}} = 7$

$\Rightarrow pq = 343$

$\therefore (p-q)^2 = (p+q)^2 - 4pq$

$= (344)^2 - 4 \cdot 343$

$= 118336 - 1372$

$= 116964$.

$\Rightarrow p - q = 342 \longrightarrow$ ④

③ + ④ $\Rightarrow$ $2p = 686 \Rightarrow p = 343$

③ − ④ $\Rightarrow$ $2q = 2 \Rightarrow q = 1$.

$\therefore x = p^{\frac{1}{3}} + q^{\frac{1}{3}}$

$= (343)^{\frac{1}{3}} + 1^{\frac{1}{3}}$

$= 7 + 1 = 8$.

Hence the eqn can be written as

$x^3 - 21x - 344 = 0$.

$\Rightarrow x^3 - 8x^2 + 8x^2 - 64x + 43x - 344 = 0$

$\Rightarrow x^2(x-8) + 8x(x-8) + 43(x-8) = 0$

$\Rightarrow (x-8)(x^2 + 8x + 43) = 0$

Either $x - 8 = 0$ or $x^2 + 8x + 43 = 0$

$\Rightarrow x = 8$

$\therefore x = \dfrac{-8 \pm \sqrt{64 - 172}}{2}$

$= \dfrac{-8 \pm \sqrt{-108}}{2}$

$= \dfrac{-8 \pm 6\sqrt{3}i}{2}$

$= -4 \pm 3\sqrt{3}i$

#

$$\begin{array}{r|l} & 344 \\ & \times 2 \\ \hline 2 & 2752 \\ 2 & 1376 \\ 2 & 688 \\ 2 & 344 \\ 2 & 172 \\ 2 & 86 \\ & 43 \end{array}$$

$64 - 43 = 21$

Ex Solve the eqn by Cardon's method

$$x^3 - 30x + 133 = 0 \longrightarrow (1)$$

Soln: Let $x = p^{\frac{1}{3}} + q^{\frac{1}{3}}$

$\Rightarrow x^3 = p + q + 3p^{\frac{1}{3}}q^{\frac{1}{3}}.x$

$\Rightarrow x^3 - 3p^{\frac{1}{3}}q^{\frac{1}{3}}x - (p+q) = 0 \longrightarrow (2)$.

Since (1) and (2) are identical

$\therefore 3p^{\frac{1}{3}}q^{\frac{1}{3}} = 30 \qquad p+q = -133 \longrightarrow (3)$

$\Rightarrow p^{\frac{1}{3}}q^{\frac{1}{3}} = 10$

$\Rightarrow pq = 1000$

$\therefore (p-q)^2 = (p+q)^2 - 4pq$

$= (-133)^2 - 4.1000$

$= 17689 - 4000$

$= 13689$

$\therefore p-q = 117 \longrightarrow (4)$.

(3) + (4) $\Rightarrow 2p = -16 \Rightarrow p = -8$, (3) - (4) $\Rightarrow 2q = -250$

$\Rightarrow q = -125$.

$\therefore x = (-8)^{\frac{1}{3}} + (-125)^{\frac{1}{3}}$

$= -2 - 5 = -7$.

Hence the eqn can be written as

$x^3 - 30x + 133 = 0$

$\Rightarrow x^3 + 7x^2 - 7x^2 - 49x + 19x + 133 = 0$

$\Rightarrow x^2(x+7) - 7x(x+7) + 19(x+7) = 0$

$\Rightarrow (x+7)(x^2 - 7x + 19) = 0$

Either $x + 7 = 0$ or $x^2 - 7x + 19 = 0$

$\Rightarrow x = -7$

$x = \dfrac{7 \pm \sqrt{49-76}}{2}$

$= \dfrac{7 \pm \sqrt{-27}}{2}$

$= \dfrac{7 \pm 3\sqrt{3}i}{2}$

$\therefore$ The roots are $-7, \dfrac{7 \pm 3\sqrt{3}i}{2}$ #