Ch: Sequences

Def^n: Function: Let A and B be two sets. If there exists a rule, denoted by 'f', which associates to each element $a \in A$, a unique element $b \in B$, we say that 'f' is a function from A into B. Symbolically, we write this as: $f: A \longrightarrow B$ or $A \xrightarrow{f} B$.

which reads 'f' is a function of A into B' or 'f maps A into B'.

A function is also called a mapping.

The element b is called the 'f-image of a' or 'the image of a under f' and is denoted by $f: a \rightarrow b$. We write $b = f(a)$ which is called the value of 'f' for a. We also say that 'a is a pre-image of b'.

Def. The set A is called the domain of f; and B the co-domain of f.

Def. Range of $f = \{f(x) : x \in A\}$, i.e., set of f-images of all the elements of the domain A.

Sequence

Def^n:- A sequence is a function whose domain is the set N of all natural numbers whereas the range may be any set S. In other words, a sequence in a set S is a rule which assigns to each natural number a unique element of S.

Real Sequence

A real sequence is a function whose domain is the set N of all natural numbers and range a subset of the set R of real numbers.

symbolically, $f: N \to R$ is a real sequence.

Note: The sequence $x: N \to R$ is denoted by $\{x_n\}$ or $\langle x_n \rangle$. $x_1, x_2 \dots$ are called the first, second.... terms of the sequence. The m^{th} and n^{th} terms x_m and x_n for $m \neq n$ are treated as distinct even if $x_m = x_n$ i.e., the terms occurring at different positions are treated as distinct terms even if they have the same value.

Range of Sequence:

The set of all distinct terms of a sequence is called its range.

Note: The number of terms of a sequence is always infinite. The range of a sequence may be a finite set.

e.g. if $x_n = (-1)^n$, then $\{x_n\} = \{-1, 1, -1, 1, \dots\}$.

The range of sequence $\{x_n\} = \{-1, 1\}$ which is a finite set.

Note: A sequence is usually denoted by writing its n^{th} term inside the brackets i.e by $\{x_n\}$. Sometimes it is denoted by writing all its terms within the brackets i.e. $\{x_1, x_2, x_3 \dots\}$.

Illustrations:

(1) $x_n = n^2 \ \forall n$ defines a sequence whose terms are $1^2, 2^2, 3^2, 4^2 \dots n^2, \dots$

(2) $x_n = \frac{1}{n} \ \forall \ n$ defines the sequence

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

(3) $x_n = \sqrt{n} \ \forall \ n$ defines the sequence

$1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}, 3, \sqrt{10} \ldots$

(4) $x_n = \frac{n}{n+1} \ \forall \ n$ defines the sequence

$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$

Bounded and Unbounded Sequence:-

Bounded above sequence:- A sequence $\{a_n\}$ is said to be bounded above if $\exists$ a real number K such that

$$a_n \leq K \ \forall \ n \in N.$$

i.e., if the range of the sequence is bounded above.

Bounded below sequence:- A sequence $\{a_n\}$ is said to be bounded below if $\exists$ a real number k such that

$$a_n \geq k \ \forall \ n \in N.$$

i.e., if the range of the sequence is bounded below.

Bounded Sequence:-

A sequence is said to be bounded if it is bounded above as well as below.

Thus a sequence $\{a_n\}$ is bounded if $\exists$ two real numbers k and K, $(k \leq K)$, such that

$$k \leq a_n \leq K \ \forall \ n \in N.$$

i.e. if the range of the sequence is bounded.

A sequence is said to be unbounded if it is not bounded.

Examples

i) The sequence $\{a_n\}$ defined by $a_n = \frac{1}{n}$ is bounded, since $0 < a_n \le 1$.

ii) The sequence $\{(-1)^n\}$ is bounded, since $-1 \le a_n \le 1 \quad \forall\, n \in N$.

iii) Every constant sequence is bounded.

* Th^m. A sequence $\{a_n\}$ is bounded iff $\exists$ a +ve real number M s.t. $|a_n| \le M \;\forall n \in N$.

Def. Least Upper Bound and Greatest Lower Bound of a Sequence :-

Least upper bound of a sequence :- If a sequence $\{a_n\}$ is bounded above, then $\exists$ a real number K_1 such that $a_n \le K_1 \;\forall\, n \in N$.

K_1 is called an upper bound of the sequence.

If $K_1 < K_2$, then $a_n < K_2 \;\forall\, n \in N$.

$\Rightarrow$ K_2 is also an upper bound of the sequence.

$\Rightarrow$ Any number $> K_1$ is also an upper bound of the sequence.

$\therefore$ if a sequence is bounded above, it has infinitely many upper bounds.

Of all the upper bounds of the sequence, if K is the least, then K is called the least upper bound (l. u. b.) of the sequence.

Greatest lower bound of a sequence:

If a sequence $\{a_n\}$ is bounded below, then $\exists$ a real number k_1 such that $k_1 \le a_n \;\forall\, n \in N$.

k_1 is called a lower bound of the sequence.

If $k_2 < k_1$, then $k_2 < a_n \;\forall\, n \in N$.

$\Rightarrow$ k_2 is also a lower bound of the sequence.

$\Rightarrow$ any number $< k_1$ is also a lower bound of the sequence.