Convergenke of Monotonic Sequences

Theorem-1 Every monotonically increasing sequence which is bounded above converge to its least upper bound.

Pf.: Let $\{a_n\}$ be a monotonically increasing sequence which is bounded above. Let u be the l.u.b. of the sequence $\{a_n\}$.

We shall show that $\{a_n\}$ converges to u.

Let $\varepsilon > 0$ be given.

Since $u - \varepsilon < u$, therefore, $u - \varepsilon$ is not an upper bound of $\{a_n\}$.

$\Rightarrow \exists$ a positive integer m such that $a_m > u - \varepsilon$.

Since $\{a_n\}$ is monotonically increasing

$$\therefore \quad a_n \geq a_m > u - \varepsilon \quad \forall\, n \geq m.$$

$$\Rightarrow \quad a_n > u - \varepsilon \quad \forall\, n \geq m \longrightarrow (1).$$

Also, u is the l.u.b. of $\{a_n\}$.

$$\Rightarrow \quad a_n \leq u + \varepsilon \quad \forall\, n \in N.$$

$$\Rightarrow \quad a_n < u + \varepsilon \quad \forall\, n \in N \longrightarrow (2).$$

from (1) and (2).

$$u - \varepsilon < a_n < u + \varepsilon \quad \forall\, n \geq m.$$

$$\Rightarrow \quad |a_n - u| < \varepsilon \quad \forall\, n \geq m.$$

$$\Rightarrow \quad \lim_{n \to \infty} a_n = u.$$

$\Rightarrow$ $\{a_n\}$ Converged to u.

Pf.

Theorem-2:- Every monotonically decreasing sequence which is bounded below converges to its greatest lower bound.

Pf:- Let $\{a_n\}$ be a monotonically decreasing sequence which is bounded below. Let l be the g.l.b. of sequence $\{a_n\}$.

We shall show that $\{a_n\}$ converges to l.

Let $\varepsilon > 0$ be given.

Since $l + \varepsilon > l$, therefore, $l + \varepsilon$ is not a lower bound of $\{a_n\}$.

$\Rightarrow \exists$ a positive integer m such that $a_m < l + \varepsilon$.

Since $\{a_n\}$ is monotonically decreasing

$$\therefore \quad a_n \leq a_m < l + \varepsilon \quad \forall\, n \geq m.$$

$$\Rightarrow \quad a_n < l + \varepsilon \quad \forall\, n \geq m \longrightarrow (1).$$

Also, l is the g.l.b. of $\{a_n\}$

$$\Rightarrow \quad a_n \geq l > l - \varepsilon \quad \forall\, n \in \mathbb{N}.$$

$$\Rightarrow \quad a_n > l - \varepsilon \quad \forall\, n \in \mathbb{N} \longrightarrow (2).$$

From (1) and (2).,

$$l - \varepsilon < a_n < l + \varepsilon \quad \forall\, n \geq m.$$

$$\Rightarrow \quad |a_n - l| < \varepsilon \quad \forall\, n \geq m.$$

$$\Rightarrow \quad \lim_{n \to \infty} a_n = l.$$

$\Rightarrow \quad \{a_n\}$ converges to l.

═══ (H).

Art. Cauchy's principles of general convergence :-
(Cauchy Convergence criterion).

A sequence $\{a_n\}$ is convergent if given $\varepsilon > 0$, there exists a positive integer m such that

$$|a_n - a_m| < \varepsilon \quad \text{whenever } n \geq m.$$

Pf :- Let the sequence $\{a_n\}$ converges to l. The given $\varepsilon > 0$, there must exist a positive integer m such that

$$|a_n - l| < \frac{\varepsilon}{2} \quad \text{whenever } n \geq m.$$

In particular, $|a_m - l| < \frac{\varepsilon}{2}$.

$$\therefore \quad |a_n - a_m| = |(a_n - l) - (a_m - l)|$$
$$\leq |a_n - l| + |a_m - l|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{whenever } n \geq m.$$

Note :- The ~~converge~~ converse of the above Thm. is also true.

Examples

(1) If $u_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$, prove that $\{u_n\}$ is monotonic increasing and bounded.

Solⁿ:- We have

$$u_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$$

$$\therefore u_{n+1} = \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2}$$

$$u_{n+1} - u_n = \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} - \frac{1}{n+2} - \cdots - \frac{1}{2n}$$

$$= \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1}$$

$$= \frac{(2n+2)(n+1) + (2n+1)(n+1) - (2n+1)(2n+2)}{(2n+1)(2n+2)(n+1)}$$

$$= \frac{\cancel{2n^2} + \cancel{2n} + \cancel{2n} + \cancel{2} + \cancel{2n^2} + \cancel{2n} + n + 1 - \cancel{4n^2} - \cancel{4n} - \cancel{2n} - \cancel{2}}{(2n+1)(2n+2)(n+1)}$$

$$= \frac{\cancel{(n+1)}}{(2n+1)(2n+2)\cancel{(n+1)}}$$

$$= \frac{1}{2(n+1)(2n+1)} > 0$$

$$\therefore u_{n+1} > u_n.$$

$\therefore \{u_n\}$ is monotonic increasing.

We have, $n > 1$

$$\Rightarrow n + n > 1 + n.$$

$$\Rightarrow 2n > 1 + n$$

$$\Rightarrow \frac{1}{2n} < \frac{1}{1+n}$$

$$\therefore \frac{1}{1+n} > \frac{1}{2n}.$$

Similarly, $\frac{1}{2+n} > \frac{1}{2n}$.

$$\therefore u_n > \frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n}$$

i.e. $u_n > \frac{n}{2n}$

i.e $u_n > \frac{1}{2}$

$\therefore \frac{1}{2} < u_n$.

Again,

$$n+2 > n+1$$

$$\Rightarrow \frac{1}{n+2} < \frac{1}{n+1}$$

$$n+3 > n+1$$

$$\Rightarrow \frac{1}{n+3} < \frac{1}{n+1}$$

$$\therefore u_n < \frac{1}{n+1} + \frac{1}{n+1} + \cdots + \frac{1}{n+1}$$

i.e. $u_n < \frac{n}{n+1}$

$$= \frac{n+1-1}{n+1}$$

$$= 1 - \frac{1}{n+1}$$

$$< 1$$

$$\therefore u_n < 1.$$

Hence $\frac{1}{2} < u_n < 1$.

$\therefore \{u_n\}$ is bounded.