11. $\frac{1}{a^2+1} + \frac{1}{2a^2+1} + \frac{1}{3a^2+1} + \cdots$

Soln:- Here $u_n = \frac{1}{na^2+1}$

Let $v_n = \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{1}{na^2+1}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{n}{na^2+1} = \lim_{n\to\infty} \frac{1}{a^2+\frac{1}{n}}$$

$$= \frac{1}{a^2} \neq 0.$$

Now let $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{n}$ which is divt. $[\because p=1]$.

$\therefore \sum_{n=1}^{\infty} u_n$ is also divt. #

13. $\frac{1}{1!} + \frac{3}{2!} + \frac{5}{3!} + \frac{7}{4!} + \cdots$

Soln:- Here $u_n = \frac{2n-1}{n!}$

Let $v_n = \frac{n}{n!} = \frac{n}{n(n-1)!} = \frac{1}{(n-1)!}$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{2n-1}{n!}}{\frac{1}{(n-1)!}} = \lim_{n\to\infty} \frac{(n-1)!\,(2n-1)}{n\,(n-1)!}$$

$$= \lim_{n\to\infty} \frac{2n-1}{n} = \lim_{n\to\infty} \left(2-\frac{1}{n}\right)$$

$$= 2 \neq 0.$$

Now $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{(n-1)!}$

$$= \frac{1}{\lfloor 0} + \frac{1}{1!} + \frac{1}{2!} + \cdots$$

$$= 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots$$

$$= e$$

$\therefore \sum_{n=1}^{\infty} v_n$ is convt. and hence $\sum_{n=1}^{\infty} u_n$ is also convt. #

14. $1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \cdots$

Soln:- Here $u_n = \frac{n^2}{n!} = \frac{n.n}{n(n-1)!} = \frac{n}{(n-1)!} = \frac{n-1+1}{(n-1)!}$

$$= \frac{n-1}{(n-1)(n-2)!} + \frac{1}{(n-1)!}$$

$$= \frac{1}{(n-2)!} + \frac{1}{(n-1)!}$$

Let $v_n = \dfrac{1}{(n-2)!}$

$$\underset{n\to\infty}{Lt} \frac{u_n}{v_n} = \underset{n\to\infty}{Lt} \frac{\frac{1}{(n-2)!} + \frac{1}{(n-1)!}}{\frac{1}{(n-2)!}}$$

$$= \underset{n\to\infty}{Lt} \frac{\frac{n^2}{n!}}{\frac{1}{(n-2)!}} = \underset{n\to\infty}{Lt} \frac{n^2 (n-2)!}{n!}$$

$$= \underset{n\to\infty}{Lt} \frac{n^2 (n-2)!}{n(n-1)(n-2)!} = \underset{n\to\infty}{Lt} \frac{n^2}{n^2(1-\frac{1}{n})}$$

$$= \underset{n\to\infty}{Lt} \frac{\cancel{n^2}\, 1}{(1-\frac{1}{n})} = 1 \neq 0.$$

$$\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{(n-2)!} = \frac{1}{\lfloor -1} + \frac{1}{\lfloor 0} + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \cdots$$

$$= 0 + 1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \cdots$$

$= e$, which is convgt.

$\therefore \sum_{n=1}^{\infty} u_n$ is also Convergent. #

09 / 15. $\dfrac{1}{1^p} + \dfrac{1}{3^p} + \dfrac{1}{5^p} + \dfrac{1}{7^p} + \cdots$

~~Test the convergence~~

or, $\cancel{\sum_{n=1}^{\infty} \frac{1}{n^p}, p > 1.}$

$= \cancel{\frac{1}{1^p} + \frac{1}{2^1}}$

Soln :- Here $u_n = \dfrac{1}{(2n-1)^p}$

Let $v_n = \dfrac{1}{n^p}$

$$\underset{n\to\infty}{Lt} \frac{u_n}{v_n} = \underset{n\to\infty}{Lt} \frac{\frac{1}{(2n-1)^p}}{\frac{1}{n^p}}$$

$$= \underset{n\to\infty}{Lt} \frac{n^p}{(2n-1)^p} = \underset{n\to\infty}{Lt} \frac{1}{(2-\frac{1}{n})^p}$$

$$= \frac{1}{2} \neq 0$$

Now $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$ is convt. for $p > 1$.

and divt. for $p < 1$ or $p = 1$. #

87, 90 / 16. $1 + \dfrac{1}{2^2} + \dfrac{2^2}{3^3} + \dfrac{3^3}{4^4} + \dfrac{4^4}{5^5} + \cdots$

08 Soln :- Here $u_n = \dfrac{n^n}{(n+1)^{n+1}}$ [neglecting first term].

Let $v_n = \dfrac{n^n}{n^{n+1}} = \dfrac{1}{n}$

$$\underset{n\to\infty}{Lt} \frac{u_n}{v_n} = \underset{n\to\infty}{Lt} \frac{\frac{n^n}{(n+1)^{n+1}}}{\frac{1}{n}} = \underset{n\to\infty}{Lt} \frac{n \cdot n^n}{(n+1)^{n+1}}$$

$$= \underset{n\to\infty}{Lt} \frac{n^{n+1}}{(n+1)^{n+1}} = \underset{n\to\infty}{Lt} \frac{1}{(1+\frac{1}{n})^{n+1}} = 1 \neq 0$$

Now $\sum_{n=1}^{\infty} V_n = \sum_{n=1}^{\infty} \frac{1}{n}$ which is divt. $(\because p=1)$

$\therefore \sum_{n=1}^{\infty} U_n$ is also divt. #

~~2005~~ ~~17~~. $\frac{1}{1+2^{-1}} + \frac{2}{1+2^{-2}} + \frac{3}{1+2^{-3}} + \dots$

Solⁿ :- Here $U_n = \frac{n}{1+2^{-n}} = \frac{n}{1+\frac{1}{2^n}} = \frac{2^n n}{1+2^n}$.

Let $V_n = \frac{n 2^n}{2^n} = n$.

$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{\frac{n \cdot 2^n}{2^n+1}}{n} = \lim_{n\to\infty} \frac{n \cdot 2^n}{n(2^n+1)} = \lim_{n\to\infty} \frac{2^n}{2^n+1}$

$= \lim_{n\to\infty} \frac{2^n}{2^n(1+\frac{1}{2^n})} = \lim_{n\to\infty} \frac{1}{1+2^{-n}} = 1 \neq 0$.

Now $\sum_{n=1}^{\infty} V_n = \sum_{n=1}^{\infty} n$, which is divt. $[\because p=1]$

$\therefore \sum_{n=1}^{\infty} U_n$ is also divt. #

* ~~18~~. $\frac{1}{\sqrt{1^2-1}} + \frac{1}{\sqrt{2^2-1}} + \frac{1}{\sqrt{3^2-1}} + \frac{1}{\sqrt{4^2-1}} + \dots + \frac{1}{\sqrt{n^2-1}} + \dots$

Solⁿ :- Here $U_n = \frac{1}{\sqrt{n^2-1}}$

Let $V_n = \frac{1}{\sqrt{n^2}} = \frac{1}{n}$.

$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^2-1}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2-1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1-\frac{1}{n^2}}}$

$= 1 \neq 0$

Now $\sum_{n=1}^{\infty} V_n = \sum_{n=1}^{\infty} \frac{1}{n}$ which is divt. $[\because p=1]$.

$\therefore \sum_{n=1}^{\infty} U_n$ is also divt. #

* 19. $\frac{1^4}{1^3} + \frac{2^4}{2^3} + \frac{3^4}{3^3} + \dots + \frac{4+10n}{n^3}$

Solⁿ :- Here $U_n = \frac{4+10n}{n^3}$

Let $V_n = \frac{n}{n^3} = \frac{1}{n^2}$.

$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{\frac{4+10n}{n^3}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{4+10n}{n}$

$= \lim_{n\to\infty} \left(10 + \frac{4}{n}\right) = 10 \neq 0$

Now $\sum_{n=1}^{\infty} V_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ which is Convt. $(\because p=2>1)$

$\therefore \sum_{n=1}^{\infty} U_n$ is also convt. #

4. $\sum u_n$, where $u_n = \sqrt{n^3+1} - \sqrt{n^3}$.

<u>Soln</u>:- we have

$$u_n = \sqrt{n^3+1} - \sqrt{n^3}$$

$$= \frac{\left(\sqrt{n^3+1} - \sqrt{n^3}\right)\left(\sqrt{n^3+1} + \sqrt{n^3}\right)}{\sqrt{n^3+1} + \sqrt{n^3}}$$

$$= \frac{1}{\sqrt{n^3+1} + \sqrt{n^3}}$$

Let $V_n = \frac{1}{\sqrt{n^3}} = \frac{1}{n\sqrt{n}}$

$$\underset{n\to\infty}{Lt} \frac{u_n}{V_n} = \underset{n\to\infty}{Lt} \frac{n\sqrt{n}}{\sqrt{n^3+1} + \sqrt{n^3}}$$

$$= \underset{n\to\infty}{Lt} \frac{\sqrt{n^3}}{\sqrt{n^3\left(1+\frac{1}{n^3}\right)} + \sqrt{n^3}}$$

$$= \underset{n\to\infty}{Lt} \frac{\sqrt{n^3}}{\sqrt{n^3}\left[1+\sqrt{1+\frac{1}{n^3}}\right]}$$

$$= \underset{n\to\infty}{Lt} \frac{1}{1+\sqrt{1+\frac{1}{n^3}}}$$

$$= \frac{1}{1+1} = \frac{1}{2} \neq 0$$

Now $\sum_{n=1}^{\infty} V_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ which is convt.

$\left[\because p = \frac{3}{2} > 1\right]$

$\therefore \sum_{n=1}^{\infty} u_n$ is also Convt.

#